arXiv:1211.0699v1 [math.RA] 4 Nov 2012

Some properties of symbol algebras of degree three

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Abstract. In this paper we will study some properties of the matrix representations of symbol algebras of degree three, we study some equations with coefficients in these algebras, we find an octonion algebra in a symbol algebra of degree three, we define the Fibonacci symbol elements and we give some properties of them.

KeyWords: symbol algebras; matrix representation; octonion algebras; Fibonacci numbers.

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2000 AMS Subject Classification: 15A24, 15A06, 16G30, 1R52, 11R37, 11B39.

0. Preliminaries

Let n be an arbitrary positive integer such that $char(K)$ does not divide n and let K be a field which contains a primitive n-th root of unity. Let $K^* = K \backslash \{0\}$, $a, b \in K^*$ and let S be the algebra over K generated by elements x and y where

$$x^n = a, y^n = b, yx = \omega xy.$$

This algebra is called a *symbol algebra* (also known as a *power norm residue algebra*) and it is denoted by $\left(\frac{a,\ b}{K,\omega}\right)$. J. Milnor, in [Mi; 71], calls it the symbol algebra because of its connection with the K-theory and with the Steinberg symbol. For $n = 2$, we obtain the quaternion algebra. For details about Steinberg symbol, the reader is referred to [La; 05].

In this paper we will study some properties of the matrix representations of symbol algebras of degree 3, we used some of these properties to solve

some equations with coefficients in these algebras and we find an octonion algebra in a symbol algebra of degree three. Since symbol algebras generalize the quaternion algebras, starting from some results given in the paper [Ho; 63], in which the author defined and studied Fibonacci quaternions, we define in the same way the Fibonacci symbol elements and we will study the properties of them.The study of symbol algebras of degree three involves very complicated calculus and, usually, can be hard to find examples for some notions. We computed the formula for the reduced norm of a Fibonacci symbol element and, using this expresion, we find an infinite set of invertible elements.

1. Introduction

In the following, we assume that K is a commutative field with $charK \neq 2,3$ and A is a finite dimensional algebra over the field K. The *center* $C(A)$ of an algebra A is the set of all elements $c \in A$ which commute and associate with all elements $x \in A$. An algebra A is a *simple* algebra if A is not a zero algebra and $\{0\}$ and A are the only ideals of A. The algebra A is called *central simple* if the algebra $A_F = F \otimes_K A$ is simple for every extension F of K. Equivalently, a central simple algebra is a simple algebra with $C(A) = K$. We remark that each simple algebra is central simple over its center. If A is a central simple algebra, then $\dim A = n = m^2$, with $m \in \mathbb{N}$. The *degree* of the central simple algebra A, denoted by $DegA$, is $DegA = m$.

If A is an algebra over the field K, a *subfield* of the algebra A is a subalgebra L of A such that L is a field. The subfield L is called a *maximal subfield* of the algebra A if there is not a subfield F of A such that $L \subset F$. If the algebra A is a central simple algebra, the subfield L of the algebra A is called a *strictly maximally subfield* of A if $[L : K] = m$, where $[L : K]$ is the degree of the extension $K \subset L$.

Let $L \subset M$ be a field extension. This extension is called a *cyclic extension* if it is a Galois extension and the Galois group $G\,(M/L)$ is a cyclic group. A central simple algebra A is called a *cyclic algebra* if there is L, a strictly maximally subfield of the algebra A, such that L/K is a cyclic extension.

Proposition 1.1. ([Pi; 82], Proposition a, p. 277) *Let $K \subset L$ be a cyclic extension with the Galois cyclic group $G = G\,(L/K)$ of order n and*

generated by the element σ. If A is a cyclic algebra and contains L as a strictly maximally subfield, then there is an element $x \in A - \{0\}$ such that:

i) $A = \bigoplus_{0 \leq j \leq n-1} x^j L$;

ii) $x^{-1}\gamma x = \sigma(\gamma)$, *for all* $\gamma \in L$;

iii) $x^n = a \in K^*$.

We will denote a cyclic algebra A with (L, σ, a).

We remark that a symbol algebra is a central simple cyclic algebra of degree n. For details about central simple algebras and cyclic algebras, the reader is referred to [Pi; 82].

Definition 1.2. [Pi; 82] Let A be an algebra over the field K. If $K \subset L$ is a finite field extension and n a natural number, then the K-algebra morphism $\varphi : A \rightarrow \mathcal{M}_n(L)$ is called a *representation of the algebra A*. The $\varphi-$*characteristic polynomial* of the element $a \in A$ is $P_\varphi(X, a) = \det(XI_n - \varphi(a))$, the $\varphi-$*norm* of the element $a \in A$ is $\eta_\varphi(a) = \det \varphi(a)$ and the $\varphi-$*trace* of the element $a \in A$ is $\tau_\varphi(a) = tr(\varphi(y))$. If A is a $K-$ central simple algebra such that $n = DegA$, then the representation φ is called a *splitting representation* of the algebra A.

Remark 1.3. i) If X,$Y \in \mathcal{M}_n(K)$, K an arbitrary field, then we know that $tr(X^t) = tr(X)$ and $tr(X^tY) = tr(XY^t)$. It results that $tr(XYX^{-1}) = tr(Y)$. Indeed,

$tr(XYX^{-1}) = tr((X^t)^tYX^{-1}) =$

$= tr(X^t(YX^{-1})^t) = tr(X^t(X^{-1})^tY^t) =$

$= tr(Y^t) = tr(Y)$.

ii) ([Pi; 82], p. 296) If $\varphi_1 : A \rightarrow \mathcal{M}_n(L_1)$, $\varphi_2 : A \rightarrow \mathcal{M}_n(L_2)$ are two splitting representations of $K-$algebra A, then $P_{\varphi_1}(X, a) = P_{\varphi_2}(X, a)$. It results that the φ_1-*characteristic polynomial* is the same with the φ_2-*characteristic polynomial,* φ_1-norm is the same with φ_2-norm, the φ_1-trace is the same with φ_2-trace and we will denote them by $P(X, a)$ instead of $P_\varphi(X, a)$, $\eta_{A/K}$ instead of η_{φ_1} or simply η and τ instead of τ_φ, when is no confusion in notation. In this case, the polynomial $P(X, a)$ is called *the characteristic polynomial,* the norm η is called the *reduced norm* of the element $a \in A$ and τ is called the *trace* of the element $a \in A$.

Proposition 1.4. ([Pi; 82], Corollary a, p. 296) *If A is a central simple algebra over the field K of degree m, $\varphi : A \rightarrow \mathcal{M}_r(L)$ is a matrix*

representation of A, then $m \mid r$, $\eta_\varphi = \eta^{r/m}$, $\tau_\varphi = (r/m)\,\tau$ and $\eta_\varphi(a), \tau_\varphi(a) \in K$, for all $a \in A$.

2. Matrix representations for the symbol algebras of degree three

Let ω be a cubic root of unity, K be a field such that $\omega \in K$ and $S = \left(\frac{a,\ b}{K,\omega}\right)$ be a symbol algebra over the field K generated by elements x and y where

$$x^3 = a, y^3 = b, yx = \omega xy, a, b \in K^*. \tag{2.1.}$$

In [Ti; 00], the author gave many properties of the left and right matrix representations for the real quaternion algebra. Using some ideas from this paper, in the following, we will study the left and right matrix representations for the symbol algebras of degree 3.

A basis in the algebra S is

$$B = \{1, x, x^2, y, y^2, xy, x^2y^2, x^2y, xy^2\}, \tag{2.2.}$$

see [Mil; 12], [Gi, Sz; 06].

Let $z \in S$,

$$z = c_0 + c_1x + c_2x^2 + c_3y + c_4y^2 + c_5xy + c_6x^2y^2 + c_7x^2y + c_8xy^2 \tag{2.3.}$$

and $\Lambda(z) \in \mathcal{M}_9(K)$ be the matrix with the columns the coefficients in K of the basis B for the elements $\{z \cdot 1, zx, zx^2, zy, zy^2, zxy, zx^2y^2, zx^2y, zxy^2\}$:

$$\Lambda(z) = \begin{pmatrix} c_0 & ac_2 & ac_1 & bc_4 & bc_3 & ab\omega^2c_6 & ab\omega^2c_5 & ab\omega c_8 & ab\omega c_7 \\ c_1 & c_0 & ac_2 & bc_8 & bc_5 & b\omega^2c_4 & ab\omega^2c_7 & ab\omega c_6 & b\omega c_3 \\ c_2 & c_1 & c_0 & bc_6 & bc_7 & b\omega^2c_8 & a\omega^2c_3 & b\omega c_4 & b\omega c_5 \\ c_3 & a\omega c_7 & a\omega^2c_5 & c_0 & bc_4 & ac_2 & ab\omega c_8 & ac_1 & ab\omega^2c_6 \\ c_4 & a\omega^2c_6 & a\omega c_8 & c_3 & c_0 & a\omega c_7 & ac_1 & a\omega^2c_5 & ac_2 \\ c_5 & \omega c_3 & a\omega^2c_7 & c_1 & bc_8 & c_0 & ab\omega c_6 & ac_2 & b\omega^2c_4 \\ c_6 & \omega^2c_8 & \omega c_4 & c_7 & c_2 & \omega c_5 & c_0 & \omega^2c_3 & c_1 \\ c_7 & \omega c_5 & \omega^2c_3 & c_2 & bc_6 & c_1 & b\omega c_4 & c_0 & b\omega^2c_8 \\ c_8 & \omega^2c_4 & a\omega c_6 & c_5 & c_1 & \omega c_3 & ac_2 & a\omega^2c_7 & c_0 \end{pmatrix}.$$

Let $\alpha_{ij} \in \mathcal{M}_3(K)$ be the matrix with 1 in position (i,j) and zero in the rest and

$$\gamma_1 = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\beta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}.$$

Proposition 2.1. *The map $\Lambda : S \to \mathcal{M}_9(K)$, $z \mapsto \Lambda(z)$ is an $K-$algebra morphism.*

Proof. For $z_1, z_2 \in S$, $d \in K$, we have that $\Lambda(z_1 + z_2) = \Lambda(z_1) + \Lambda(z_2)$ and $\Lambda(dz_1) = d\Lambda(z_1)$.

With the above notations, let $\Lambda(x) = X = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \alpha_{31} & a\beta_2 \\ 0 & \beta_1 & \alpha_{13} \end{pmatrix} \in \mathcal{M}_9(K)$ and

$$\Lambda(y) = Y = \begin{pmatrix} 0 & b\alpha_{12} & \omega b\beta_4 \\ \beta_3 & \alpha_{21} & 0 \\ \omega^2\alpha_{23} & \omega\alpha_{33} & \omega^2\alpha_{12} \end{pmatrix}.$$

By straightforward calculations, we obtain:
$\Lambda(x^2) = X^2, \Lambda(y^2) = Y^2, \Lambda(xy) = XY,$
$\Lambda(x^2y) = \Lambda(x^2)\Lambda(y) = \Lambda(x)\Lambda(xy) = X^2Y,$
$\Lambda(xy^2) = \Lambda(x)\Lambda(y^2) = \Lambda(xy)\Lambda(y) = XY^2,$
$\Lambda(x^2y^2) = \Lambda(x^2)\Lambda(y^2) = \Lambda(x)\Lambda(xy^2) = \Lambda(x^2y)\Lambda(y) = X^2Y^2.$

Therefore, we have $\Lambda(z_1z_2) = \Lambda(z_1)\Lambda(z_2)$. It results that Λ is a $K-$algebra morphism. □

The morphism Λ is called the *left matrix representation* for the algebra S.

Definition 2.2. For $Z \in S$, we denote by $\overrightarrow{Z} = (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)^t \in \mathcal{M}_{9\times 1}(K)$ *the vector representation* of the element Z.

Proposition 2.3. *Let $Z, A \in S$, then:*
i) $\overrightarrow{Z} = \Lambda(Z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *where $0 \in \mathcal{M}_{8\times 1}(K)$ is the zero matrix.*
ii) $\overrightarrow{AZ} = \Lambda(A)\overrightarrow{Z}$.

Proof. ii) From i), we obtain that
$\overrightarrow{AZ} = \Lambda\left(AZ\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Lambda\left(A\right)\Lambda\left(Z\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Lambda\left(A\right)\overrightarrow{Z}.\Box$

Remark 2.4. 1) We remark that an element $z \in S$ is an invertible element in S if and only if $\det\Lambda\left(z\right) \neq 0$.

2) The $\Lambda-$*norm* of the element $z \in S$ is $\eta_{\Lambda}\left(z\right) = \det\Lambda\left(z\right) = \eta^{3}\left(z\right)$ and $\tau_{\Lambda}\left(z\right) = 9tr\Lambda\left(z\right)$. Indeed, from Proposition 1.4, if $A = S, K = L, m = 3, r = 9, \varphi = \Lambda$, we obtain the above relation.

3) We have that $\tau_{\Lambda}\left(z\right) = tr\Lambda\left(z\right) = 9c_{0}$.

Let $z \in S, z = A + By + Cy^{2}$, where $A = c_{0} + c_{1}x + c_{2}x^{2}, B = c_{3} + c_{5}x + c_{7}x^{2}, C = c_{4} + c_{8}x + c_{6}x^{2}$. We denote by $z_{\omega} = A + \omega By + \omega^{2}Cy^{2}, z_{\omega^{2}} = A + \omega^{2}By + \omega Cy^{2}$.

Proposition 2.5. *Let* $S = \left(\frac{1,1}{K,\omega}\right)$, *therefore* $\eta_{\Lambda}\left(z\right) = \eta_{\Lambda}\left(z_{\omega}\right) = \eta_{\Lambda}\left(z_{\omega^{2}}\right)$.

Proof. The left matrix representation for the element $z \in S$ is $\Lambda\left(z\right) =$

$$\begin{pmatrix}
c_0 & c_2 & c_1 & c_4 & c_3 & \omega^2 c_6 & \omega^2 c_5 & \omega c_8 & \omega c_7 \\
c_1 & c_0 & c_2 & c_8 & c_5 & \omega^2 c_4 & \omega^2 c_7 & \omega c_6 & \omega c_3 \\
c_2 & c_1 & c_0 & c_6 & c_7 & \omega^2 c_8 & \omega^2 c_3 & \omega c_4 & \omega c_5 \\
c_3 & \omega c_7 & \omega^2 c_5 & c_0 & c_4 & a c_2 & \omega c_8 & c_1 & \omega^2 c_6 \\
c_4 & \omega^2 c_6 & \omega c_8 & c_3 & c_0 & \omega c_7 & c_1 & \omega^2 c_5 & c_2 \\
c_5 & \omega c_3 & \omega^2 c_7 & c_1 & c_8 & c_0 & \omega c_6 & c_2 & \omega^2 c_4 \\
c_6 & \omega^2 c_8 & \omega c_4 & c_7 & c_2 & \omega c_5 & c_0 & \omega^2 c_3 & c_1 \\
c_7 & \omega c_5 & \omega^2 c_3 & c_2 & c_6 & c_1 & \omega c_4 & c_0 & \omega^2 c_8 \\
c_8 & \omega^2 c_4 & \omega c_6 & c_5 & c_1 & \omega c_3 & c_2 & \omega^2 c_7 & c_0
\end{pmatrix}$$

and for the element z_{ω} is

$$\Lambda\left(z_{\omega}\right) = \begin{pmatrix}
c_0 & c_2 & c_1 & \omega^2 c_4 & \omega c_3 & \omega c_6 & c_5 & c_8 & \omega^2 c_7 \\
c_1 & c_0 & c_2 & \omega^2 c_8 & \omega c_5 & \omega c_4 & c_7 & c_6 & \omega^2 c_3 \\
c_2 & c_1 & c_0 & \omega^2 c_6 & \omega c_7 & \omega c_8 & c_3 & c_4 & \omega^2 c_5 \\
c_3 & \omega^2 c_7 & c_5 & c_0 & \omega^2 c_4 & c_2 & c_8 & c_1 & \omega c_6 \\
c_4 & \omega c_6 & c_8 & \omega c_3 & c_0 & \omega^2 c_7 & c_1 & c_5 & c_2 \\
c_5 & \omega^2 c_3 & c_7 & c_1 & \omega^2 c_8 & c_0 & c_6 & c_2 & \omega c_4 \\
c_6 & \omega c_8 & c_4 & \omega c_7 & c_2 & \omega^2 c_5 & c_0 & c_3 & c_1 \\
c_7 & \omega^2 c_5 & c_3 & c_2 & \omega^2 c_6 & c_1 & c_4 & c_0 & \omega c_8 \\
c_8 & \omega c_4 & c_6 & \omega c_5 & c_1 & \omega^2 c_3 & c_2 & c_7 & c_0
\end{pmatrix}.$$

Denoting by $D_{rs} = \left(d_{ij}^{rs}\right) \in \mathcal{M}_9(K)$ the matrix defined such that $d_{kk}^{rs} = 1$ for $k \notin \{r, s\}$, $d_{rr}^{rs} = d_{ss}^{rs} = 0$, $d_{rs}^{rs} = d_{sr}^{rs} = 1$ and zero in the rest, we have that $\det D_{rs} = -1$. If we multiply a matrix A to the left with D_{rs}, the new matrix is obtained from A by changing the line r with the line s and if we multiply a matrix A to the right with D_{rs}, the new matrix is obtained from A by changing the column r with the column s. By straightforward calculations, it results that $\Lambda(z_\omega) = D_{79}D_{48}D_{46}D_{57}D_{21}D_{23}\Lambda(z)D_{12}D_{23}D_{49}D_{79}D_{48}D_{59}$, therefore $\eta_\Lambda(z) = \det \Lambda(z) = \det \Lambda(z_\omega) = \eta_\Lambda(z_\omega)$. In the same way, we get that $\eta_\Lambda(z) = \det \Lambda(z) = \det \Lambda(z_{\omega^2}) = \eta_\Lambda(z_{\omega^2})$.$\square$

Similar to the matrix $\Lambda(z), z \in S$, we define $\Gamma(z) \in \mathcal{M}_9(K)$ to be the matrix with the columns the coefficients in K of the basis B for the elements $\{z \cdot 1, xz, x^2z, yz, y^2z, xyz, x^2y^2z, x^2yz, xy^2z\}$. This matrix is

$$\Gamma(z) = \begin{pmatrix} c_0 & ac_2 & ac_1 & bc_4 & bc_3 & ab\omega^2c_6 & ab\omega^2c_5 & ab\omega c_8 & ab\omega c_7 \\ c_1 & c_0 & ac_2 & b\omega c_8 & b\omega^2c_5 & bc_4 & ab\omega c_7 & ab\omega^2c_6 & bc_3 \\ c_2 & c_1 & c_0 & b\omega^2c_6 & b\omega c_7 & b\omega c_8 & ac_3 & bc_4 & b\omega^2c_5 \\ c_3 & ac_7 & ac_5 & c_0 & bc_4 & a\omega^2c_2 & ab\omega^2c_8 & a\omega c_1 & ab\omega c_6 \\ c_4 & ac_6 & ac_8 & c_3 & c_0 & a\omega^2c_7 & a\omega^2c_1 & a\omega c_5 & a\omega c_2 \\ c_5 & c_3 & ac_7 & \omega c_1 & b\omega^2c_8 & c_0 & ab\omega c_6 & a\omega^2c_2 & bc_4 \\ c_6 & c_8 & c_4 & \omega^2c_7 & \omega c_2 & \omega c_5 & c_0 & c_3 & \omega^2c_1 \\ c_7 & c_5 & c_3 & \omega^2c_2 & b\omega c_6 & \omega c_1 & bc_4 & c_0 & b\omega^2c_8 \\ c_8 & c_4 & ac_6 & \omega c_5 & \omega^2c_1 & c_3 & a\omega c_2 & a\omega^2c_7 & c_0 \end{pmatrix}$$

Let $\alpha_{ij} \in \mathcal{M}_3(K)$ be the matrix with 1 in position (i, j) and zero in the rest and

$$\beta_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega & 0 \end{pmatrix}, \beta_6 = \begin{pmatrix} 0 & 1 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\beta_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \beta_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proposition 2.6. *With the above notations, we have:*
i) $\Gamma(d_1z_1 + d_2z_2) = d_1\Gamma(z_1) + d_2\Gamma(z_2)$, *for all* $z_1, z_2 \in S$ *and* $d_1, d_1 \in K$.
ii) $\Gamma(z_1z_2) = \Gamma(z_2)\Gamma(z_1)$, *for all* $z_1, z_2 \in S$.

Proof. Let $\Gamma(x) = U = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \omega\alpha_{31} & a\omega\beta_6 \\ 0 & \omega\beta_5 & \omega^2\alpha_{13} \end{pmatrix} \in \mathcal{M}_9(K)$ and

$\Gamma\left(y\right)=V=\left(\begin{array}{ccc}0 & b\alpha_{12} & \omega b\beta_{8}\\ \beta_{7} & \alpha_{21} & 0\\ \alpha_{23} & \alpha_{33} & \alpha_{12}\end{array}\right).$

By straightforward calculations, we obtain:
$\Gamma\left(x^{2}\right)=U^{2},\Gamma\left(y^{2}\right)=V^{2},\Gamma\left(yx\right)=UV,$
$\Gamma\left(x^{2}y\right)=\Gamma\left(y\right)\Gamma\left(x^{2}\right)=\Gamma(xy)\Gamma(x)=VU^{2},$
$\Gamma\left(xy^{2}\right)=\Gamma\left(y^{2}\right)\Gamma\left(x\right)=\Gamma(y)\Gamma(xy)=V^{2}U,$
$\Gamma\left(x^{2}y^{2}\right)=\Gamma\left(y^{2}\right)\Gamma\left(x^{2}\right)=\Gamma(xy^{2})\Gamma(x)=\Gamma(y)\Gamma(x^{2}y)=V^{2}U^{2}.$

Therefore, we have that $\Gamma\left(z_{1}z_{2}\right)=\Gamma\left(z_{2}\right)\Gamma\left(z_{1}\right)$ and Γ is a $K-$algebra morphism.□

Proposition 2.7. *Let $Z,A\in S$, then:*
i) $\overrightarrow{Z}=\Gamma\left(Z\right)\left(\begin{array}{c}1\\ 0\end{array}\right)$, *where $0\in\mathcal{M}_{8\times1}\left(K\right)$ is the zero matrix.*
ii) $\overrightarrow{ZA}=\Gamma\left(A\right)\overrightarrow{Z}$.
iii) $\Lambda\left(A\right)\Gamma\left(B\right)=\Gamma\left(B\right)\Lambda\left(A\right).$

Proof. ii) From i), we obtain
$\overrightarrow{ZA}=\Gamma\left(ZA\right)\left(\begin{array}{c}1\\ 0\end{array}\right)=\Gamma\left(A\right)\Gamma\left(Z\right)\left(\begin{array}{c}1\\ 0\end{array}\right)=\Gamma\left(A\right)\overrightarrow{Z}.$

iii) Using left matrix representation, it results that
$\overrightarrow{AZB}=\Lambda\left(A\right)\overrightarrow{ZB}=\Lambda\left(A\right)\Gamma\left(B\right)\overrightarrow{Z}.$
Using right matrix representation, we have
$\overrightarrow{AZB}=\Gamma\left(B\right)\overrightarrow{AZ}=\Gamma\left(B\right)\Lambda\left(A\right)\overrightarrow{Z}.$□

Proposition 2.8. *Let $S=\left(\frac{1,1}{K,\omega}\right)$, therefore $\eta_{\Gamma}\left(z\right)=\eta_{\Gamma}\left(z_{\omega}\right)=\eta_{\Gamma}\left(z_{\omega^{2}}\right).$*

Proof. Similar with the proof of Proposition 2.5. □

Theorem 2.9. *Let $S=\left(\frac{a,b}{K,\omega}\right)$ be a symbol algebra of degree three and*
$M_{9}=\left(1,\frac{1}{a}x,\frac{1}{a}x^{2},\frac{1}{b}y,\frac{1}{b}y^{2},\frac{1}{ab}xy,\frac{1}{ab}x^{2}y^{2},\frac{1}{ab}x^{2}y,\frac{1}{ab}xy^{2}\right),$
$N_{9}=\left(1,x^{2},x,y^{2},y,x^{2}y^{2},xy,xy^{2},x^{2}y\right)^{t}$
$M_{10}=\left(1,\frac{1}{a}x^{2},\frac{1}{a}x,\frac{1}{b}y^{2},\frac{1}{b}y,\frac{1}{ab}x^{2}y^{2},\frac{1}{ab}xy,\frac{1}{ab}xy^{2},\frac{1}{ab}x^{2}y\right),$
$N_{10}=\left(1,x,x^{2},y,y^{2},xy,x^{2}y^{2},x^{2}y,xy^{2}\right)^{t}$. *The following relation is true*

$$M_{9}\Lambda\left(z\right)N_{9}=M_{10}\Gamma^{t}\left(z\right)N_{10}=3z,z\in S.$$

Proof. We will prove that $M_{10}\Gamma^t(z)N_{10}=3z$. Indeed, computing $M_{10}\Gamma^t(z)$, we obtain
$(c_0+c_1x+c_2x^2+c_3y+c_4y^2+\omega^2c_5xy+\omega^2c_6x^2y^2+\omega c_7x^2y+\omega c_8xy^2,$
$c_1+c_2x+\frac{1}{a}c_0x^2+\omega^2c_5y+\omega c_8y^2+\omega c_7xy+\frac{1}{a}c_6x^2y^2+\frac{1}{a}c_3x^2y+\omega^2c_6xy^2,$
$c_2+\frac{1}{a}c_0x+\frac{1}{a}c_1x^2+\omega c_7y+\omega^2c_4y^2+\frac{1}{a}c_5xy+\frac{1}{a}\omega c_8x^2y^2+\frac{1}{a}\omega^2c_5x^2y+\frac{1}{a}c_4xy^2,$
$c_3+c_5x+c_7x^2+c_4y+\frac{1}{b}\omega c_0y^2+\omega^2c_8xy+\frac{1}{b}\omega^2c_2x^2y^2+\omega c_6x^2y+\frac{1}{b}\omega c_8xy^2,$
$c_4+c_8x+c_6x^2+\frac{1}{b}c_0y+\frac{1}{b}c_3y^2+\frac{1}{b}\omega^2c_1xy+\frac{1}{b}\omega^2c_7x^2y^2+\frac{1}{b}\omega c_2x^2y+\frac{1}{b}\omega c_5xy^2,$
$c_5+c_7x+\frac{1}{a}c_3x^2+\omega^2c_8y+\frac{1}{a}\omega c_1y^2+\omega c_6xy+\frac{1}{ab}c_0x^2y^2+\frac{1}{a}c_4x^2y+\frac{1}{b}\omega^2c_2xy^2,$
$c_6+\frac{1}{a}c_4x+\frac{1}{a}c_8x^2+\frac{1}{b}\omega c_2y+\frac{1}{b}\omega^2c_7y^2+\frac{1}{ab}c_0xy+\frac{1}{ab}\omega c_5x^2y^2+\frac{1}{ab}\omega^2c_1x^2y+\frac{1}{ab}c_3xy^2,$
$c_7+\frac{1}{a}c_3x+\frac{1}{a}c_5x^2+\omega c_6y+\omega^2\frac{1}{b}c_2y^2+\frac{1}{a}c_4xy+\frac{1}{ab}\omega c_1x^2y^2+\frac{1}{a}\omega^2c_8x^2y+\frac{1}{ab}c_0xy^2,$
$c_8+c_6x+\frac{1}{a}c_8x^2+\frac{1}{b}\omega^2c_1y+\frac{1}{b}\omega c_5y^2+\frac{1}{b}\omega c_2xy+\frac{1}{ab}c_3x^2y^2+\frac{1}{a}c_0x^2y+\frac{1}{b}\omega^2c_7xy^2).$
Therefore, computing $M_{10}\Gamma^t(z)N_{10}$, we obtain
$c_0+c_1x+c_2x^2+c_3y+c_4y^2+\omega^2c_5xy+\omega^2c_6x^2y^2+\omega c_7x^2y+\omega c_8xy^2+$
$c_0+c_1x+c_2x^2+\omega c_3y+\omega^2c_4y^2+c_5xy+\omega c_6x^2y^2+\omega^2c_7x^2y+c_8xy^2+$
$c_0+c_1x+c_2x^2+\omega^2c_3y+\omega c_4y^2+\omega c_5xy+c_6x^2y^2+c_7x^2y+\omega^2c_8xy^2+$
$c_0+\omega c_1x+\omega^2c_2x^2+c_3y+c_4y^2+c_5xy+\omega c_6x^2y^2+c_7x^2y+\omega^2c_8xy^2+$
$c_0+\omega^2c_1x+\omega c_2x^2+c_3y+c_4y^2+\omega c_5xy+c_6x^2y^2+\omega^2c_7x^2y+c_8xy^2+$
$\omega^2c_0+c_1x+\omega c_2x^2+c_3y+\omega c_4y^2+c_5xy+\omega^2c_6x^2y^2+c_7x^2y+c_8xy^2+$
$\omega^2c_0+\omega c_1x+c_2x^2+\omega c_3y+c_4y^2+\omega^2c_5xy+c_6x^2y^2+c_7x^2y+c_8xy^2+$
$\omega c_0+c_1\omega^2x+c_2x^2+c_3y+\omega^2c_4y^2+c_5xy+c_6x^2y^2+c_7x^2y+\omega c_8xy^2+$
$\omega c_0+c_1x+\omega^2c_2x^2+\omega^2c_3y+c_4y^2+c_5xy+c_6x^2y^2+\omega c_7x^2y+c_8xy^2=$
$=3\left(c_0+c_1x+c_2x^2+c_3y+c_4y^2+c_5xy+c_6x^2y^2+c_7x^2y+c_8xy^2\right)=3z$, since $\omega^2+\omega+1=0.\square$

3. Some equations with coefficients in a symbol algebra of degree three

Using some properties of left and right matix representations found in the above section, we solve some equations with coefficients in a symbol algebra of degree three.

Let S be an associative algebra of degree three. For $z\in S$, let $P(X,z)$ be the characteristic polynomial for the element a

$$P(X,z)=X^3-\tau(z)X^2+\pi(z)X-\eta(z)\cdot 1, \tag{3.1.}$$

where τ is a linear form, π is a quadratic form and η a cubic form.

Proposition 3.1. ([Fa; 88], Lemma) *With the above notations, denoting by* $z^* = z^2 - \tau(z) z + \pi(z) \cdot 1$, *for an associative algebra of degree three, we have:*

i) $\pi(z) = \tau(z^*)$.
ii) $2\pi(z) = \tau(z)^2 - \tau(z^2)$.
iii) $\tau(zw) = \tau(wz)$.
iv) $z^{**} = \eta(z) z$.
v) $(zw)^* = w^* z^*$.
vi) $\pi(zw) = \pi(wz)$.□

In the following, we will solve some equations with coefficients in the symbol algebra of degree three $S = \left(\frac{a,b}{K,\omega}\right)$. For each element $Z \in S$ relation (3.1) holds. First, we remark that if the element $Z \in S$ has $\eta(Z) \neq 0$, then Z is an invertible element. Indeed, from $(3.1.)$, we have that $ZZ^* = \eta(Z)$, therefore $Z^{-1} = \frac{Z}{\eta(Z)}$.

We consider the following equations:

$$AZ = ZA \tag{3.2.}$$

$$AZ = ZB \tag{3.3.}$$

$$AZ - ZA = C \tag{3.4.}$$

$$AZ - ZB = C, \tag{3.5.}$$

with $A, B, C \in S$.

Proposition 3.2. *i) Equation* (3.2) *has non-zero solutions in the algebra* S.

ii) If equation (3.3) *has nonzero solutions* Z *in the algebra* S *such that* $\eta(Z) \neq 0$, *then* $\tau(A) = \tau(B)$ *and* $\eta(A) = \eta(B)$.

iii) If equation (3.4) *has solution, then this solution is not unique.*

iv) If $\Lambda(A) - \Gamma(B)$ *is an invertible matrix, then equation* (3.5) *has a unique solution.*

Proof. i) Using vector representation, we have that $\overrightarrow{AZ} = \overrightarrow{ZA}$. It results that $\Lambda(A) \overrightarrow{Z} = \Gamma(A) \overrightarrow{Z}$, therefore $(\Lambda(A) - \Gamma(A)) \overrightarrow{Z} = 0$. The matrix $\Lambda(A) - \Gamma(A)$ has the determinant equal with zero (first column is zero), then equation (3.2) has non-zero solutions.

ii) We have that $\eta(AZ) = \eta(BZ)$ and $\eta(A) = \eta(B)$ since Z is an invertible element in S. Using representation Λ, we obtain $\Lambda(A) \Lambda(Z) =$

$\Lambda(Z)\Lambda(B)$. From Remark 1.3 i), it results that $\Lambda(A)=\Lambda(Z)\Lambda(B)(\Lambda(Z))^{-1}$. Therefore $\tau(A)=tr(\Lambda(A))=tr(\Lambda(Z)\Lambda(B)(\Lambda(Z))^{-1})=tr(\Lambda(B))=\tau(B)\in K$.

iii) Using the vector representation, we have $(\Lambda(A)-\Gamma(A))\overrightarrow{Z}=C$ and, since the matrix $\Lambda(A)-\Gamma(A)$ has the determinant equal with zero (first column is zero), if equation (3.4) has solution, then this solution is not unique.

iv) Using vector representation, we have $(\Lambda(A)-\Gamma(B))\overrightarrow{Z}=C.\square$

Proposition 3.3. *Let*
$A=a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy+a_6x^2y^2+a_7x^2y+a_8xy^2\in S$,
$B=b_0+b_1x+b_2x^2+b_3y+b_4y^2+b_5xy+b_6x^2y^2+b_7x^2y+b_8xy^2\in S, A_0=A-a_0\neq 0$ *and* $B_0=B-b_0\neq 0$. *If* $a_0=b_0, A_0\neq -B_0, \eta(A_0)=\eta(B_0)=0$ *and* $\pi(A_0)=\pi(B_0)\neq 0$, *therefore all solutions of equation* (3.3) *are in the* K*-algebra* $\mathcal{A}(A,B)$, *the subalgebra of* S *generated by the elements* A *and* B, *and have the form* $\lambda_1X_1+\lambda_2X_2$, *where* $\lambda_1,\lambda_2\in K$, $X_1=A_0+B_0$ *and* $X_2=\pi(A_0)-A_0B_0$.

Proof. First, we remark that X_1 and X_2 are solutions of the equation (3.3). Indeed, since $A_0^2=B_0^2=-\pi(A_0)=-\pi(B_0)$, we obtain:
$AX_1-X_1B=A(A-a_0+B-b_0)-(A-a_0+B-b_0)B=$
$=A^2-Aa_0+AB-b_0A-AB+a_0B-B^2+b_0B=$
$=A^2-Aa_0-b_0A+a_0B-B^2+b_0B=$
$=(A_0+a_0)^2-a_0(A_0+a_0)-b_0(A_0+a_0)+a_0(B_0+b_0)-(B_0+b_0)^2+b_0(B_0+b_0)=$
$=A_0^2+a_0^2+2a_0A_0-a_0A_0-a_0^2-b_0A_0-b_0a_0+a_0B_0+a_0b_0-B_0^2-b_0^2-2b_0B_0+b_0B_0+b_0^2=0.$
$AX_2-X_2B=A(\pi(A_0)-A_0B_0)-(\pi(A_0)-A_0B_0)B=$
$=(A_0+a_0)(\pi(A_0)-A_0B_0)-(\pi(A_0)-A_0B_0)(B_0+b_0)=$
$=A_0\pi(A_0)-A_0^2B_0+a_0\pi(A_0)-a_0A_0B_0-\pi(A_0)B_0-b_0\pi(A_0)+A_0B_0^2+b_0A_0B_0=0.$

We prove that X_1 and X_2 are linearly independent elements. If $\alpha_1X_1+\alpha_2X_2=0$, it results that $\alpha_2\pi(A_0)=0$, therefore $\alpha_2=0$. We obtain that $\alpha_1=0$. Obviously, each element of the form $\lambda_1X_1+\lambda_2X_2$, where $\lambda_1,\lambda_2\in K$ is a solution of the equation (3.3) and since $\pi(A_0)\neq 0$ we have that $\mathcal{A}(A,B)=\mathcal{A}(X_1,X_2)$. Since each solution of the equation (3.3) belongs to algebra $\mathcal{A}(A,B)$, it results that all solutions have the form $\lambda_1X_1+\lambda_2X_2$, where $\lambda_1,\lambda_2\in K.\square$

4. Octonion algebra in a symbol algebra of degree three

An associative finite dimensional K-algebra A is *semisimple* if it can be expressed as a finite and unique direct sum of simple algebras. An associative K-algebra A is *separable* if for every field extension $K \subset L$ the algebra $A \otimes_K L$ is semisimple. We have that any central simple algebra is a separable algebra over its center (see [Ha; 00], p.463). A *composition* algebra A is a unital (not necessarily associative) algebra over K together with a nondegenerate quadratic form N which satisfies $N(xy) = N(x) N(y), x, y \in A$.

Theorem 4.1. ([Ja; 81], Theorem 6.2.3) *Let A be a finite-dimensional algebra with unity over the field K and $\varphi : A \to K$ be a nondegenerate quadratic form such that $\varphi(xy) = \varphi(x) \varphi(y)$ for all $x, y \in A$. Then the algebra A has dimension $1, 2, 4$ or 8. If $\dim A \in \{4.8\}$, A is a quaternion or an octonion algebra.* $\Box$

Let $S = \left(\frac{a,b}{K,\omega}\right)$ be a symbol algebra of degree n. For $n = 2$, we obtain the quaternion algebra. For $n = 3$, the obtained symbol algebra has dimension 9 over the field K and, since an octonion algebra generalizes the quaternion algebra and has dimension 8 less than 9, we ask if we can find a relation between a symbol algebra of degree three and an octonion algebra.

Proposition 4.2. ([Fa; 88], Theorem) *If A is an associative algebra of degree three over a field K containing the cubic root of unity, then, using notations from Proposition 3.1, the quadratic form π permits compositions $\pi(z \circ w) = \pi(z) \pi(w)$ on $\widetilde{A} = \{u \in A \;/\; \tau(u) = 0\}$ relative to the product*

$$z \circ w = \omega z w - \omega^2 w z - \frac{2\omega + 1}{3} \tau(zw) \cdot 1, z, w \in \widetilde{A}.$$

If A is separable over K, therefore the quadratic form π is nondegenerate and we can find a new product "$\bigtriangledown$" on $(\widetilde{A}, \circ)$ such that $\left(\widetilde{A}, \bigtriangledown\right)$ is a composition algebra. $\Box$

Since π is a nondegenerate quadratic form on S, it is also nondegenerate on $\widetilde{S}$, then there is an element $u \in \widetilde{S}$ such that $\pi(u) \neq 0$. Using some ideas given in [Ka; 53], let $v = \frac{u^2}{\pi(u)}$.

Proposition 4.3. *The linear maps* $R_v^\circ : \widetilde{S} \to \widetilde{S}, R_v^\circ (x) = x \circ v$ *and* $L_v^\circ : \widetilde{S} \to \widetilde{S}, L_v^\circ (x) = x \circ v$ *are bijective.*

Proof. Let $R_v^\circ : \widetilde{S} \to \widetilde{S}, R_v^\circ (x) = x \circ v$. Since $\pi (R_v^\circ (x)) = \pi (x \circ v) = \pi (x) \pi (v) = \pi (x)$, if $R_v^\circ (x) = 0$ it results that $\pi (x) = 0$, Using that π is nondegenerate, we obtain $x = 0$, therefore R_v° is bijective. $\square$

From the above proposition, on $\widetilde{S}$ we define a new multiplication

$$z \bigtriangledown w = (R_v^{\circ -1} (z)) \circ (L_v^{\circ -1} (w)), w, z \in \widetilde{S}.$$

We have that $v \circ v$ is the unity element and $\pi (z \bigtriangledown w) = \pi (z) \pi (w)$. Indeed, it results
$z \bigtriangledown (v \circ v) = (R_v^{\circ -1} (z)) \circ (L_v^{\circ -1} (v \circ v)) =$
$= (R_v^{\circ -1} (z)) \circ L_v^{\circ -1} (L_v^\circ (v)) =$
$= (R_v^{\circ -1} (z)) \circ v = R_v^\circ (R_v^{\circ -1} (z)) = z =$
$= (v \circ v) \bigtriangledown z$ and
$\pi (z \bigtriangledown w) = \pi ((R_v^{\circ -1} (z)) \circ (L_v^{\circ -1} (w))) =$
$= \pi (R_v^{\circ -1} (z)) \pi (L_v^{\circ -1} (w)) =$
$= \pi (z) \pi (w)$, since $\pi (z) = \pi (R_v^\circ (R_v^{\circ -1} (z))) = \pi (R_v^{\circ -1} (z))$. Therefore the algebra $\left(\widetilde{S}, \bigtriangledown\right)$ is a composition algebra of dimension 8 and, from Theorem 4.1, we obtain that $\left(\widetilde{S}, \bigtriangledown\right)$ is an octonion algebra with the norm π. This algebra is a not a division algebra since, from Proposition 3.1. and relation (2.2), we have $0 = \tau (\eta (x) x^*) = \tau ((x^*)^+) = \pi (x^*)$, for the element $x \in \widetilde{S}$.

From the above, we proved the following theorem:

Theorem 4.3. *Let* $S = \left(\frac{a,b}{K,\omega}\right)$ *be a symbol algebra of degree* 3. *On the vector space* $\widetilde{S}$, *we define the following products:*

$$z \circ w = \omega z w - \omega^2 w z - \frac{2\omega + 1}{3} \tau (zw) \cdot 1, z, w \in \widetilde{S}$$

and

$$z \bigtriangledown w = (R_v^{\circ -1} (z)) \circ (L_v^{\circ -1} (w)), w, z \in \widetilde{S}.$$

Therefore in a symbol algebra of degree three we can always find an octonion non-division algebra, namely $\left(\widetilde{S}, \bigtriangledown\right)$.$\square$

In the above Theorem, we provided an example of associative algebra of degree three in which we can find an octonion non-division algebra, other than the example gave in [Fa; 88].

5. Fibonacci symbol elements

In this section we will introduce the Fibonacci symbol elements and we eill compute the reduced norm of such an element. This relation will helps us to find an infinite set of invertible elements. First of all, we recall and give some properties of Fibonacci number, properties which will be used in our proofs.

Fibonacci numbers are the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ...,$$

with the nth term given by the formula:

$$f_n = f_{n-1} + f_{n-2,}\ n \geq 2,$$

where $f_0 = 0, f_1 = 1$. The expresion for the nth term is

$$f_n = \frac{1}{\sqrt{5}}[\alpha^n - \beta^n], \tag{5.1.}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Proposition 5. 1. *Let* $(f_n)_{n\geq 0}$ *be the Fibonacci sequence* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $(\forall)$ $n \in \mathbb{N}$. *Then*

$$f_n + f_{n+3} = 2f_{n+2}, (\forall) n \in \mathbb{N}.$$

Proof. We will prove by induction over $n \in \mathbb{N}$ the following statement:

$$P(n) : f_n + f_{n+3} = 2f_{n+2}.$$

We verify that $P(0)$ and $P(1)$ are true.
We suppose that $P(k)$ is true for any $k \leq n - 1$ and we prove that $P(n)$ is

true. Using the induction hypothesis, we have:

$$f_n + f_{n+3} = f_{n-1} + f_{n-2} + f_{n+2} + f_{n+1} =$$
$$= (f_{n-1} + f_{n+2}) + (f_{n-2} + f_{n+1}) = 2f_{n+1} + 2f_n = 2f_{n+2}.$$

□

Proposition 5. 2. *Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $(\forall)$ $n\in \mathbb{N}$. Then*

$$f_n + f_{n+4} = 3f_{n+2}, (\forall) n \in \mathbb{N}.$$

Proof. By induction over $n \in \mathbb{N}$.□

Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $(\forall)$ $n\in \mathbb{N}$. Let ω be a primitive root of unity of order 3, let $K = \mathbb{Q}(\omega)$ be the cyclotomic field and let $S = \left(\frac{a,b}{K,\omega}\right)$ be the symbol algebra of degree 3. Thus, S has an K- basis $\{x^i y^j | 0 \leq i, j < 3\}$ such that $x^3 = a\in K^*$, $y^3 = b\in K^*$, $yx = \omega xy$

Let $z\in S$, $z = \sum\limits_{i,j=1}^{n} x^i y^j c_{ij}$. The reduced norm of z is

$$\eta(z) = a^2 \cdot \left(c_{20}^3 + bc_{21}^3 + b^2 c_{22}^3 - 3bc_{20}c_{21}c_{22}\right) +$$
$$+a \cdot \left(c_{10}^3 + bc_{11}^3 + b^2 c_{12}^3 - 3bc_{10}c_{11}c_{12}\right) -$$
$$-3a \cdot \left(c_{00}c_{10}c_{20} + bc_{01}c_{11}c_{21} + b^2 c_{02}c_{12}c_{22}\right) -$$
$$-3ab\omega \left(c_{00}c_{12}c_{21} + c_{01}c_{10}c_{22} + c_{02}c_{11}c_{20}\right) -$$
$$-3ab\omega^2 \left(c_{00}c_{11}c_{22}+c_{02}c_{10}c_{21}+c_{01}c_{12}c_{20}\right) + \left(c_{00}^3+bc_{01}^3+b^2c_{02}^3\text{-}3bc_{00}c_{01}c_{02}\right) \tag{5.2.}$$

(See [Pi; 82], p.299).

We define the *n th Fibonacci symbol element* to be the element

$$F_n = f_n \cdot 1 + f_{n+1} \cdot x + f_{n+2} \cdot x^2 + f_{n+3} \cdot y+$$
$$+ f_{n+4} \cdot xy + f_{n+5} \cdot x^2 y + f_{n+6} \cdot y^2 + f_{n+7} \cdot xy^2 + f_{n+8} \cdot x^2 y^2. \tag{5.3.}$$

In [Ho,61] A.F. Horadam generalized the Fibonacci numbers. The generalized Fibonacci numbers are:

$$h_n = h_{n-1} + h_{n-2},\ n \in \mathbb{N}, n \geq 2,$$

$h_0 = p, h_1 = q$, where p, q are arbitrary integers. These numbers satify the equality $h_{n+1} = pf_n + qf_{n+1}$, $(\forall)n \in \mathbb{N}$. In the following will be easy for use to use instead of h_{n+1} the notation $h_{n+1}^{p,q}$. It is obviulusly that

$$h_n^{p,q} + h_n^{p',q'} = h_n^{p+p',q+q'},$$

for n a positive integers number and p, q, p', q' integers numbers. (See [Fl, Sh; 12])

Remark 5.3. ([Cu; 76]). *Let* $(f_n)_{n\geq 0}$ *be the Fibonacci sequence* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $(\forall)$ $n \in \mathbb{N}$. *Then*
i)

$$f_n^2 + f_{n-1}^2 = f_{2n-1}, (\forall)n \in \mathbb{N}^*;$$

ii)

$$f_{n+1}^2 - f_{n-1}^2 = f_{2n}, (\forall)n \in \mathbb{N}^*;$$

iii)

$$f_{n+3}^2 = 2f_{n+2}^2 + 2f_{n+1}^2 - f_n^2, (\forall)n \in \mathbb{N};$$

iv)

$$f_n^2 - f_{n-1}f_{n+1} = (-1)^{n-1}, (\forall)n \in \mathbb{N}^*.$$

We define the *nth generalized Fibonacci symbol element* to be the element

$$H_n^{p,q} = h_n^{p,q} \cdot 1 + h_{n+1}^{p,q} \cdot x + h_{n+2}^{p,q} \cdot x^2 + h_{n+3}^{p,q} \cdot y +$$

$$+ h_{n+4}^{p,q} \cdot xy + h_{n+5}^{p,q} \cdot x^2y + h_{n+6}^{p,q} \cdot y^2 + h_{n+7}^{p,q} \cdot xy^2 + h_{n+8}^{p,q} \cdot x^2y^2. \qquad (5.4.)$$

What algebraic strucure have the set of Fibonacci symbol elements or the set of generalized Fibonacci symbol? The answer will be found in the following proposition.

Proposition 5.4. *Let* M *the set*

$$M = \left\{ \sum_{i=1}^{n} H_{n_i}^{p_i,q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall)i = \overline{1,n} \right\} \cup \{0\}$$

is a $\mathbb{Z}-$ module.

Proof. The proof is imediately if we remark that for $n_1, n_2 \in \mathbb{N}$, $p_1, p_2, \alpha_1, \alpha_2 \in \mathbb{Z}$, we have:

$$\alpha_1 h_{n_1}^{p_1,q_1} + \alpha_2 h_{n_2}^{p_2,q_2} = h_{n_1}^{\alpha_1 p_1, \alpha_1 q_1} + h_{n_2}^{\alpha_2 p_2, \alpha_2 q_2}.$$

Therefore, we obtain that M is a $\mathbb{Z}-$ submodule of the symbol algebra $A.\square$

In the followng we will compute the reduced norm for the nth Fibonacci symbol element.

Proposition 5.5. *Let F_n be the nth Fibonacci symbol element. Let ω be a primitive root of order 3 of unity and let $K = \mathbb{Q}(\omega)$ be the cyclotomic field. Then the norm of F_n is*

$$\begin{aligned} \eta(F_n) = 4a^2 h_{n+3}^{211,14} \cdot \left(h_{2n}^{84,135} - 2f_n^2\right) + 8h_{n+3}^{8,3} \cdot \left(h_{2n}^{12,20} - f_n^2\right) + \\ +a[f_{n+2}\left(\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533}\right) + \\ f_{n+3} \cdot \left(\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163}\right) - \\ -f_n^2 \left(\omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523}\right) - \\ -(-1)^n \left(\omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138}\right)]. \end{aligned}$$

Proof. In this proof, we denote with $E(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. We obtain:

$$\eta(F_n) = \eta(F_n) + 3 \cdot \left(1 + \omega + \omega^2\right) \cdot \left(f_n^3 + f_{n+1}^3 + f_{n+2}^3 + ... + f_{n+8}^3\right) =$$

$$\begin{aligned} = a^2 \cdot \left(f_{n+2}^3 + f_{n+5}^3 + f_{n+8}^3 - 3f_{n+2}f_{n+5}f_{n+8}\right) + \\ +a \cdot \left(f_{n+1}^3 + f_{n+4}^3 + f_{n+7}^3 - 3f_{n+1}f_{n+3}f_{n+7}\right) + \\ +a \cdot \left(f_n^3 + f_{n+1}^3 + f_{n+2}^3 - 3f_n f_{n+1} f_{n+2}\right) + \\ +a \cdot \left(f_{n+3}^3 + f_{n+4}^3 + f_{n+5}^3 - 3f_{n+3}f_{n+4}f_{n+5}\right) + \\ +a \cdot \left(f_{n+6}^3 + f_{n+7}^3 + f_{n+8}^3 - 3f_{n+6}f_{n+7}f_{n+8}\right) + \\ +a \cdot \left(\omega^3 f_n^3 + f_{n+5}^3 + f_{n+7}^3 - 3\omega f_n f_{n+5} f_{n+7}\right) + \\ +a \cdot \left(f_{n+1}^3 + f_{n+3}^3 + \omega^3 f_{n+8}^3 - 3\omega f_{n+1} f_{n+3} f_{n+8}\right) + \end{aligned}$$

$$+a\cdot\left(f_{n+2}^3+f_{n+4}^3+\omega^3 f_{n+6}^3-3\omega f_{n+2}f_{n+4}f_{n+6}\right)+$$
$$+a\cdot\left(f_n^3+f_{n+4}^3+\omega^6 f_{n+8}^3-3\omega^2 f_n f_{n+4}f_{n+8}\right)+$$
$$+a\cdot\left(\omega^6 f_{n+1}^3+f_{n+5}^3+f_{n+6}^3-3\omega^2 f_{n+1}f_{n+5}f_{n+6}\right)+$$
$$a\cdot\left(f_{n+2}^3+f_{n+3}^3+\omega^6 f_{n+7}^3-3\omega^2 f_{n+2}f_{n+3}f_{n+7}\right)+\left(f_n^3+f_{n+3}^3+f_{n+6}^3-3f_n f_{n+3}f_{n+6}\right)=$$
$$=a^2\cdot E\left(f_{n+2},f_{n+5},f_{n+8}\right)+a\cdot E\left(f_{n+1},f_{n+4},f_{n+7}\right)+$$
$$+a\cdot E\left(f_n,f_{n+1},f_{n+2}\right)+a\cdot E\left(f_{n+3},f_{n+4},f_{n+5}\right)+$$
$$+a\cdot E\left(f_{n+6},f_{n+7},f_{n+8}\right)+a\cdot E\left(\omega f_n,f_{n+5},f_{n+7}\right)+$$
$$+a\cdot E\left(f_{n+1},f_{n+3},\omega f_{n+8}\right)+a\cdot E\left(f_{n+2},f_{n+4},\omega f_{n+6}\right)+$$
$$+a\cdot E\left(f_n,f_{n+4},\omega^2 f_{n+8}\right)+a\cdot E\left(\omega^2 f_{n+1},f_{n+5},f_{n+6}\right)+$$
$$+a\cdot E\left(f_{n+2},f_{n+3},\omega^2 f_{n+7}\right)+E\left(f_n,f_{n+3},f_{n+6}\right).$$

Now, we compute $E\left(f_{n+2},f_{n+5},f_{n+8}\right)$.

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)=$$
$$=\frac{1}{2}\left(f_{n+2}+f_{n+5}+f_{n+8}\right)\left[\left(f_{n+5}-f_{n+2}\right)^2+\left(f_{n+8}-f_{n+5}\right)^2+\left(f_{n+8}-f_{n+2}\right)^2\right].$$

Using Proposition 5.1, Proposition 5.2, Remark 5.3 (iii) and the recurrence of the Fibonacci sequence, we obtain:

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)=$$
$$=\frac{1}{2}\left(2f_{n+4}+f_{n+8}\right)\left[\left(f_{n+4}+f_{n+1}\right)^2+\left(f_{n+7}+f_{n+4}\right)^2+\left(f_{n+1}+2f_{n+4}+f_{n+7}\right)^2\right]=$$
$$=\frac{1}{2}\left(f_{n+4}+3f_{n+6}\right)\cdot\left[\left(2f_{n+3}\right)^2+\left(2f_{n+6}\right)^2+\left(2f_{n+3}+2f_{n+6}\right)^2\right]$$
$$=2\left(4f_{n+4}+3f_{n+5}\right)\cdot\left(f_{n+3}^2+f_{n+6}^2+4f_{n+5}^2\right)=2\left(4f_{n+4}+3f_{n+5}\right)\cdot\left(2f_{n+4}^2+6f_{n+5}^2\right)=$$
$$=4\left(11f_{n+2}+14f_{n+3}\right)\cdot\left(135f_{n+1}^2+82f_n^2-51f_{n-1}^2\right). \qquad (5.5.)$$

Then, we have:

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)=4\left(11f_{n+2}+14f_{n+3}\right)\cdot\left(135f_{n+1}^2+82f_n^2-51f_{n-1}^2\right). \qquad (5.6.)$$

Using Remark 5.3 (i,ii), the definition of the generalized Fibonacci sequence we have:

$E\left(f_{n+2}, f_{n+5}, f_{n+8}\right) = 4\left(11 f_{n+2} + 14 f_{n+3}\right) \cdot \left[135\left(f_{n+1}^2 - f_{n-1}^2\right) + 84\left(f_{n-1}^2 + f_n^2\right) - 2 f_n^2\right] =$

$4\left(11 f_{n+2} + 14 f_{n+3}\right) \cdot \left(135 f_{2n} + 84 f_{2n-1} - 2 f_n^2\right) = 4\left(11 f_{n+2} + 14 f_{n+3}\right) \cdot \left(h_{2n}^{84,135} - 2 f_n^2\right) =$
$4 h_{n+3}^{11,14} \cdot \left(h_{2n}^{84,135} - 2 f_n^2\right).$
Therefore, we obtain

$$E\left(f_{n+2}, f_{n+5}, f_{n+8}\right) = 4 h_{n+3}^{11,14} \cdot \left(h_{2n}^{84,135} - 2 f_n^2\right). \tag{5.7.}$$

Replacing $n \to n-1$ in relation (5.6) and using Remark 5.3, (iii) and the recurrence of the Fibonacci sequence, we obtain:

$$E\left(f_{n+1}, f_{n+4}, f_{n+7}\right) = 4\left(3 f_{n+2} + 11 f_{n+3}\right) \cdot \left(51 f_{n+1}^2 + 33 f_n^2 - 20 f_{n-1}^2\right). \tag{5.8.}$$

Now, we calculate $E\left(f_n, f_{n+1}, f_{n+2}\right)$.

$$E\left(f_n, f_{n+1}, f_{n+2}\right) =$$
$$= \frac{1}{2}\left(f_n + f_{n+1} + f_{n+2}\right) \cdot \left[\left(f_{n+1} - f_n\right)^2 + \left(f_{n+2} - f_{n+1}\right)^2 + \left(f_{n+2} - f_n\right)^2\right] =$$
$$= f_{n+2} \cdot \left(f_{n-1}^2 + f_n^2 + f_{n+1}^2\right).$$

So, we obtain

$$E\left(f_n, f_{n+1}, f_{n+2}\right) = f_{n+2} \cdot \left(f_{n+1}^2 + f_n^2 + f_{n-1}^2\right). \tag{5.9.}$$

Replacing $n \to n+3$ in relation (5.9.), using Remark 5.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

$$E\left(f_{n+3}, f_{n+4}, f_{n+5}\right) = \left(f_{n+2} + 2 f_{n+3}\right) \cdot \left(23 f_{n+1}^2 + 15 f_n^2 - 9 f_{n-1}^2\right). \tag{5.10.}$$

Replacing $n \to n+3$ in relation (5.10), using Remark 5.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

$$E\left(f_{n+6}, f_{n+7}, f_{n+8}\right) = \left(3 f_{n+2} + 4 f_{n+3}\right) \cdot \left(635 f_{n+1}^2 + 387 f_n^2 - 239 f_{n-1}^2\right). \tag{5.11.}$$

Adding equalities (5.8),(5.9),(5.10),(5.11), after straightforward calculation, we have:

$$E\left(f_{n+1},f_{n+4},f_{n+7}\right)+E\left(f_{n},f_{n+1},f_{n+2}\right)+E\left(f_{n+3},f_{n+4},f_{n+5}\right)+E\left(f_{n+6},f_{n+7},f_{n+8}\right)=$$

$$f_{n+2}\cdot\left(2511f_{n+1}^2+1573f_n^2-965f_{n-1}^2\right)+f_{n+3}\cdot\left(4790f_{n+1}^2+3030f_n^2-1854f_{n-1}^2\right). \tag{5.12.}$$

Using Remark 5.3 (i, ii), we have:

$$E\left(f_{n+1},f_{n+4},f_{n+7}\right)+E\left(f_{n},f_{n+1},f_{n+2}\right)+E\left(f_{n+3},f_{n+4},f_{n+5}\right)+E\left(f_{n+6},f_{n+7},f_{n+8}\right)=$$

$$f_{n+2}\cdot\left(2511f_{2n+1}-965f_{2n-1}+27f_n^2\right)+f_{n+3}\cdot\left(4790f_{2n+1}-1854f_{2n-1}+94f_n^2\right)=$$

$$f_{n+2}\cdot\left(1546f_{2n+1}+965f_{2n}+27f_n^2\right)+f_{n+3}\cdot\left(2936f_{2n+1}+1854f_{2n-1}+94f_n^2\right)=$$

$$=f_{n+2}\cdot\left(h_{2n+1}^{965,1546}+27f_n^2\right)+f_{n+3}\cdot\left(h_{2n+1}^{1854,2936}+94f_n^2\right)=$$

$$=f_{n+2}\cdot h_{2n+1}^{965,1546}+f_{n+3}\cdot h_{2n+1}^{1854,2936}+27f_n^2\cdot\left(f_{n+4}+67f_{n+3}\right)=$$

$$=f_{n+2}\cdot h_{2n+1}^{965,1546}+f_{n+3}\cdot h_{2n+1}^{1854,2936}+27f_n^2\cdot h_{n+4}^{67,1}.$$

So, we obtained that:

$$E\left(f_{n+1},f_{n+4},f_{n+7}\right)+E\left(f_{n},f_{n+1},f_{n+2}\right)+E\left(f_{n+3},f_{n+4},f_{n+5}\right)+E\left(f_{n+6},f_{n+7},f_{n+8}\right)=$$

$$=f_{n+2}\cdot h_{2n+1}^{965,1546}+f_{n+3}\cdot h_{2n+1}^{1854,2936}+27f_n^2\cdot h_{n+4}^{67,1}. \tag{5.13.}$$

Replacing $n\to n-1$ in relation (5.8), using Remark 5.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

$$E\left(f_n,f_{n+3},f_{n+6}\right)=8\left(8f_{n+2}+3f_{n+3}\right)\cdot\left(20f_{n+1}^2+11f_n^2-8f_{n-1}^2\right). \tag{5.14.}$$

Using Remark 5.3 (i,ii), the definition of the generalized Fibonacci sequence, we have:

$$E\left(f_n,f_{n+3},f_{n+6}\right)=8\left(8f_{n+2}+3f_{n+3}\right)\cdot\left[20\left(f_{n+1}^2-f_{n-1}^2\right)+12\left(f_{n-1}^2+f_n^2\right)-f_n^2\right]=$$

$$8\left(8f_{n+2}+3f_{n+3}\right)\cdot\left(20f_{2n}+12f_{2n-1}-f_n^2\right)=8h_{n+3}^{8,3}\cdot\left(h_{2n}^{12,20}-f_n^2\right).$$

Therefore

$$E\left(f_n,f_{n+3},f_{n+6}\right)=8h_{n+3}^{8,3}\cdot\left(h_{2n}^{12,20}-f_n^2\right). \tag{5.15}$$

Now, we compute $E\left(\omega f_n,f_{n+5},f_{n+7}\right)$.

$$E\left(\omega f_n,f_{n+5},f_{n+7}\right)=\frac{1}{2}\left(\omega f_n+f_{n+5}+f_{n+7}\right)\cdot\left[\left(f_{n+7}-f_{n+5}\right)^2+\left(f_{n+5}-\omega f_n\right)^2+\left(f_{n+7}-\omega f_n\right)^2\right].$$

By repeatedly using of Fibonacci sequence recurrence and Remark 5.3 (iii), we obtain:

$$E\left(\omega f_n, f_{n+5}, f_{n+7}\right) = \frac{1}{2}\left[2\left(2+\omega\right)f_{n+2} + \left(7-\omega\right)f_{n+3}\right]\cdot$$

$$\cdot\left[104f_{n+1}^2 + 65f_n^2 - 40f_{n-1}^2 + \left(5f_{n+1} + (3-\omega)f_n\right)^2 + \left(13f_{n+1} + (8-\omega)f_n\right)^2\right] =$$

$$= \frac{1}{2}\left[2\left(2+\omega\right)f_{n+2} + \left(7-\omega\right)f_{n+3}\right]\cdot$$

$$\cdot\left[104f_{n+1}^2 + 65f_n^2 - 40f_{n-1}^2 + \left((8-\omega)f_{n+1} + (\omega-3)f_{n-1}\right)^2 + \left((11-\omega)f_{n+1} + (\omega-8)f_{n-1}\right)^2\right].$$

In the same way, we have:

$$E\left(\omega f_n, f_{n+5}, f_{n+7}\right) = \frac{1}{2}\left[2\left(2+\omega\right)f_{n+2} + \left(7-\omega\right)f_{n+3}\right]\cdot$$

$$\cdot\left[\left(-40\omega+287\right)f_{n+1}^2 + \left(-24\omega+31\right)f_{n-1}^2 + \left(-64\omega+285\right)f_n^2 - 4\left(16\omega-55\right)\cdot(-1)^n\right]. \quad (5.16.)$$

Now, we calculate $E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right)$.

$$E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right) = \frac{1}{2}\left(f_{n+1} + f_{n+3} + \omega f_{n+8}\right)\cdot$$

$$\left[\left(f_{n+3} - f_{n+1}\right)^2 + \left(\omega f_{n+8} - f_{n+3}\right)^2 + \left(\omega f_{n+8} - f_{n+1}\right)^2\right].$$

By repeatedly using of Fibonacci sequence recurrence and Remark 5.3 (iii), we obtain:

$$E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right) = \frac{1}{2}\left[(5\omega\text{-}1)f_{n+2} + 2\left(4\omega\text{-}1\right)f_{n+3}\right]\cdot$$

$$\cdot\left[2f_{n+1}^2 + 2f_n^2\text{-}f_{n-1}^2 + \left((21\omega\text{-}2)f_{n+1} + (13\omega\text{-}1)f_n\right)^2 + \left(13\omega f_n + (21\omega\text{-}1)f_{n+1}\right)^2\right] =$$

$$= \frac{1}{2}\left[\left(5\omega-1\right)f_{n+2} + 2\left(4\omega-1\right)f_{n+3}\right]\cdot$$

$$\cdot\left[2f_{n+1}^2 + 2f_n^2\text{-}f_{n-1}^2 + \left((34\omega-3)f_{n+1}\text{-}((13\omega\text{-}1)f_{n-1}\right)^2 + \left((34\omega\text{-}1)f_{n+1}\text{-}13\omega f_{n-1}\right)^2\right].$$

Using the recurrence of the Fibonacci sequence and Remark 2 (iv) and making some calculations we have:

$$E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right) = \left[\left(5\omega-1\right)f_{n+2} + 2\left(4\omega-1\right)f_{n+3}\right]\cdot$$

$$\cdot\left[\text{-}2\left(696\omega+578\right)f_{n+1}^2\text{-}\left(182\omega+169\right)f_{n-1}^2 + 2\left(485\omega+441\right)f_n^2 + \left(970\omega+881\right)\cdot(\text{-}1)^n\right]. \quad (5.17.)$$

Now, we calculate $E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right)$.

$$E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right) =$$
$$\frac{1}{2}\left(f_{n+2}+f_{n+4}+\omega f_{n+6}\right) \cdot \left[\left(f_{n+4}\text{-}f_{n+2}\right)^2 + \left(\omega f_{n+6}\text{-}f_{n+4}\right)^2 + \left(\omega f_{n+6}\text{-}f_{n+2}\right)^2\right].$$

In the same way, we obtain:

$$E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right) = \frac{1}{2}\left[2\left(\omega+1\right) f_{n+2} + \left(3\omega+1\right) f_{n+3}\right] \cdot$$
$$\cdot \left[6f_{n+1}^2 + 3f_n^2 - 2f_{n-1}^2 + \left((8\omega\text{-}3)f_{n+1} + (5\omega\text{-}2)f_n\right)^2 + \left((8\omega\text{-}1)f_{n+1} + (5\omega-1)f_n\right)^2\right] =$$
$$= \frac{1}{2}\left[2\left(\omega+1\right) f_{n+2} + \left(3\omega+1\right) f_{n+3}\right] \cdot$$
$$\cdot \left[6f_{n+1}^2 + 3f_n^2\text{-}2f_{n-1}^2 + \left((13\omega\text{-}5)f_{n+1}\text{-}(5\omega\text{-}2)f_{n-1}\right)^2 + \left((13\omega\text{-}2)f_{n+1}\text{-}(5\omega\text{-}1)f_{n-1}\right)^2\right].$$

Using the recurrence of the Fibonacci sequence and Remark 5.3 (iv), we obtain:

$$E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right) = \frac{-1}{2}\left[2\left(\omega+1\right) f_{n+2} + \left(3\omega+1\right) f_{n+3}\right] \cdot$$
$$\cdot \left[\left(520\omega+309\right) f_{n+1}^2 + \left(80\omega+47\right) f_{n-1}^2 + \left(112\omega-21\right) f_n^2 + 8\left(14\omega+3\right) \cdot (-1)^n\right]. \tag{5.18.}$$

Adding relations (5.16), (5.17), (5.18) and making some calculus, it results:

$$E\left(\omega f_n, f_{n+5}, f_{n+7}\right) + E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right) + E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right) =$$
$$f_{n+2}\left[\left(10138\omega+6127\right) f_{n+1}^2\text{-}\left(1210\omega+5231\right) f_n^2 + \left(301\omega+1132\right) f_{n-1}^2\text{-}\,(\text{-}1)^n\left(1235\omega+5315\right)\right] +$$
$$\frac{1}{2} f_{n+3}[\left(8206\omega+27088\right) f_{n+1}^2\text{-}\left(6296\omega+17132\right) f_n^2 + \left(614\omega+3542\right) f_{n-1}^2\text{-}$$
$$- (-1)^n\left(22792\omega+32558\right)] =$$
$$f_{n+2}\left[\left(10138\omega+6127\right) f_{n+1}^2\text{-}\left(1210\omega+5231\right) f_n^2 + \left(301\omega+1132\right) f_{n-1}^2\text{-}\,(\text{-}1)^n\left(1235\omega+5315\right)\right] +$$
$$f_{n+3}[\left(4103\omega+13544\right) f_{n+1}^2\text{-}\left(3148\omega+8566\right) f_n^2 + \left(307\omega+1771\right) f_{n-1}^2\text{-}$$
$$- (-1)^n\left(11396\omega+16279\right)] \tag{5.19.}$$

Now, we compute $E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)$.

$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right) =$$

$$\frac{1}{2}\left(f_n + f_{n+4}+\omega^2 f_{n+8}\right)\cdot\left[\left(f_{n+4}-f_n\right)^2+\left(\omega^2 f_{n+8}-f_{n+4}\right)^2+\left(\omega^2 f_{n+8}-f_n\right)^2\right].$$

Using Proposition 5.2 and the recurrence of the Fibonacci sequence, we obtain:

$$f_n+f_{n+4}=3f_{n+2},\, f_{n+4}-f_n=3f_{n+1}+f_n,\, f_{n+8}=5f_{n+2}+8f_{n+3}.$$

It results:

$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)=\frac{1}{2}\left[(-5\omega+8)\, f_{n+2}+8\,(-\omega+1)\, f_{n+3}\right]\cdot$$

$$\left[\left(3f_{n+1}+f_n\right)^2+\left((21\omega^2-3)f_{n+1}+(13\omega^2-2)f_n\right)^2+\left(21\omega^2 f_{n+1}+(13\omega^2-1)f_n\right)^2\right]=$$

$$=\frac{1}{2}\left[(-5\omega+8)\, f_{n+2}+8\,(-\omega+1)\, f_{n+3}\right]\cdot$$

$$\left[\left(4f_{n+1}\text{-}f_{n-1}\right)^2+\left((34\omega^2\text{-}5)f_{n+1}\text{-}(13\omega^2\text{-}2)f_{n-1}\right)^2+\left((34\omega^2\text{-}1)f_{n+1}\text{-}(13\omega^2\text{-}1)f_{n-1}\right)^2\right].$$

Using Remark 5.3. (iv), we have:

$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)=\left[(-5\omega+8)\, f_{n+2}+8\,(-\omega+1)\, f_{n+3}\right]\cdot$$

$$\left[(1460\omega+325)\, f_{n+1}^2+(208\omega+42)\, f_{n-1}^2-(1030\omega+127)\, f_n^2-(1030\omega+127)\cdot(-1)^n\right]. \tag{5.20.}$$

Now, we compute $E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)$.

$$E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)=$$

$$\frac{1}{2}\left(f_{n+2}+f_{n+3}+\omega^2 f_{n+7}\right)\cdot\left[\left(f_{n+3}-f_{n+2}\right)^2+\left(\omega^2 f_{n+7}-f_{n+3}\right)^2+\left(\omega^2 f_{n+7}-f_{n+2}\right)^2\right].$$

Using the recurrence of the Fibonacci sequence, it results:

$$f_{n+7}=3f_{n+2}+5f_{n+3}, \omega^2 f_{n+7}-f_{n+3}=\left(8\omega^2-1\right) f_n+\left(13\omega^2-2\right) f_{n+1},$$

$\omega^2 f_{n+7}-f_{n+2}=\left(8\omega^2-1\right) f_n+\left(13\omega^2-1\right) f_{n+1}$.

Making some calculations, we obtain:

$$E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)=$$

$$\frac{-1}{2}\left[(3\omega+2)\, f_{n+2}+(5\omega+4)\, f_{n+3}\right]\cdot$$

$$\left[f_{n+1}^2+\left((21\omega^2\text{-}3)f_{n+1}\text{-}(8\omega^2\text{-}1)f_{n-1}\right)^2+\left((21\omega^2\text{-}2)f_{n+1}\text{-}(8\omega^2\text{-}1)f_{n-1}\right)^2\right]=$$

$$-\left[(3\omega+2)\,f_{n+2}+(5\omega+4)\,f_{n+3}\right]\cdot$$

$$\left[\left(441\omega^2+987\omega+553\right)f_{n+1}^2+\left(64\omega^2+144\omega+81\right)f_{n-1}^2-\left(336\omega^2+754\omega+423\right)f_{n-1}f_{n+1}\right].$$

Using Remark 5.3 (iv), we obtain:

$$E\left(f_{n+2},f_{n+3},\omega^2 f_{n+7}\right)=$$

$$-\left[(3\omega+2)\,f_{n+2}+(5\omega+4)\,f_{n+3}\right]\cdot$$

$$\left[(546\omega+112)\,f_{n+1}^2+(80\omega+17)\,f_{n-1}^2\text{-}\,(418\omega+87)\,f_n^2\text{-}\,(418\omega+87)\cdot(-1)^n\right]. \tag{5.21.}$$

Now, we compute $E\left(f_{n+5},f_{n+6},\omega^2 f_{n+1}\right).$

$$E\left(f_{n+5},f_{n+6},\omega^2 f_{n+1}\right)=\frac{1}{2}\left(f_{n+5}+f_{n+6}+\omega^2 f_{n+1}\right)\cdot$$

$$\cdot\left[\left(f_{n+6}-f_{n+5}\right)^2+\left(f_{n+5}-\omega^2 f_{n+1}\right)^2+\left(f_{n+6}-\omega^2 f_{n+1}\right)^2\right].$$

Using the recurrence of the Fibonacci sequence, we have:

$$f_{n+5}+f_{n+6}+\omega^2 f_{n+1}=(\omega+4)\,f_{n+2}+(-\omega+4)\,f_{n+3};\,f_{n+6}-f_{n+5}=3f_{n+1}+2f_n,$$

$$f_{n+5}-\omega^2 f_{n+1}=(\omega+6)\,f_{n+1}+3f_n;\,f_{n+6}-\omega^2 f_{n+1}=(\omega+9)\,f_{n+1}+4f_n.$$

It results:

$$E\left(f_{n+5},f_{n+6},\omega^2 f_{n+1}\right)=\frac{1}{2}\left[(\omega+4)\,f_{n+2}+(-\omega+4)\,f_{n+3}\right]\cdot$$

$$\cdot\left[(3f_{n+1}+2f_n)^2+((\omega+6)\,f_{n+1}+3f_n)^2+((\omega+9)\,f_{n+1}+5f_n)^2\right]=$$

$$=\frac{1}{2}\left[(\omega+4)\,f_{n+2}+(-\omega+4)\,f_{n+3}\right]\cdot$$

$$\cdot\left[(5f_{n+1}-2f_{n-1})^2+((\omega+9)\,f_{n+1}-3f_{n-1})^2+((\omega+14)\,f_{n+1}-5f_{n-1})^2\right]=$$

$$=\frac{1}{2}\left[(\omega+4)\,f_{n+2}+(-\omega+4)\,f_{n+3}\right]\cdot\left[(46\omega+302)\,f_{n+1}^2+38f_{n-1}^2-(12\omega+214)\,f_{n-1}f_{n+1}\right].$$

Using Remark 5.3 (iv), we obtain:

$$E\left(f_{n+5},f_{n+6},\omega^2 f_{n+1}\right)=\left[(\omega+4)\,f_{n+2}+(-\omega+4)\,f_{n+3}\right]\cdot$$

$$\cdot\left[(23\omega+151)\,f_{n+1}^{2}+19f_{n-1}^{2}-(6\omega+107)\,f_{n}^{2}-(6\omega+107)\cdot(-1)^{n}\right]. \tag{5.22.}$$

Adding equalities (5.20), (5.21), (5.22), we have:

$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)+E\left(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}\right)+E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)=f_{n+2}\cdot$$
$$\left[(17785\omega+11895)\,f_{n+1}^{2}\text{-}\,(13037\omega+7667)\,f_{n}^{2}+(2542\omega+1658)\,f_{n-1}^{2}\text{-}\,(13037\omega+7667)\cdot(\text{-}1)^{n}\right]$$
$$+f_{n+3}\cdot[(-2650\omega-6171)\,f_{n+1}^{2}-(15052\omega+7859)\,f_{n}^{2}+(2608\omega+2048)\,f_{n-1}^{2}-$$
$$-(15052\omega+7859)\cdot(-1)^{n}]. \tag{5.23.}$$

Adding (5.12) (5.19),(5.23), it results:

$$E\left(f_{n+1}, f_{n+4}, f_{n+7}\right)+E\left(f_n, f_{n+1}, f_{n+2}\right)+E\left(f_{n+3}, f_{n+4}, f_{n+5}\right)+E\left(f_{n+6}, f_{n+7}, f_{n+8}\right)+$$
$$E\left(\omega f_n, f_{n+5}, f_{n+7}\right)+E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right)+E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right)+$$
$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)+E\left(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}\right)+E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)=$$
$$f_{n+2}\cdot\left(2511f_{n+1}^{2}+1573f_{n}^{2}-965f_{n-1}^{2}\right)+f_{n+3}\cdot\left(4790f_{n+1}^{2}+3030f_{n}^{2}-1854f_{n-1}^{2}\right)+$$
$$f_{n+2}\cdot[(27923\omega+18022)\,f_{n+1}^{2}-(14247\omega+12898)\,f_{n}^{2}+(2843\omega+2790)\,f_{n-1}^{2}-$$
$$-(14272\omega+12928)\cdot(-1)^{n}]+f_{n+3}\cdot[(1453\omega+7373)\,f_{n+1}^{2}-(18200\omega+16425)\,f_{n}^{2}+$$
$$+(2915\omega+3819)\,f_{n-1}^{2}-(26448\omega+24138)\cdot(-1)^{n}]. \tag{5.24.}$$

Using Remark 5.3 (i,ii) and the definition of the generalized Fibonacci numbers, we obtain:

$$E\left(f_{n+1}, f_{n+4}, f_{n+7}\right)+E\left(f_n, f_{n+1}, f_{n+2}\right)+E\left(f_{n+3}, f_{n+4}, f_{n+5}\right)+E\left(f_{n+6}, f_{n+7}, f_{n+8}\right)+$$
$$E\left(\omega f_n, f_{n+5}, f_{n+7}\right)+E\left(f_{n+1}, f_{n+3}, \omega f_{n+8}\right)+E\left(f_{n+2}, f_{n+4}, \omega f_{n+6}\right)+$$
$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right)+E\left(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}\right)+E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right)=$$
$$f_{n+2}\cdot[(27923\omega+20533)\,f_{n+1}^{2}-(14247\omega+11325)\,f_{n}^{2}+(2843\omega+1825)\,f_{n-1}^{2}+$$
$$(14272\omega+12928)\cdot(\text{-}1)^{n+1}]+f_{n+3}\cdot[(1453\omega+12163)\,f_{n+1}^{2}\text{-}\,(18200\omega+13395)\,f_{n}^{2}+$$
$$+(2915\omega+1965)\,f_{n-1}^{2}+(26448\omega+24138)\cdot(-1)^{n+1}]=$$
$$f_{n+2}\cdot[(27923\omega+20533)\left(f_{n+1}^{2}\text{-}f_{n-1}^{2}\right)+(30766\omega+22358)\left(f_{n-1}^{2}+f_{n}^{2}\right)\text{-}\,(45013\omega+33683)\,f_{n}^{2}+$$
$$(14272\omega+12928)\cdot(\text{-}1)^{n+1}]+f_{n+3}\cdot[(1453\omega+12163)\left(f_{n+1}^{2}\text{-}f_{n-1}^{2}\right)+(4368\omega+14128)\left(f_{n-1}^{2}+f_{n}^{2}\right)$$
$$-(22568\omega+27523)\,f_{n}^{2}+(26448\omega+24138)\cdot(-1)^{n+1}]=f_{n+2}\cdot[(27923\omega+20533)\,f_{2n}+$$

$(30766\omega + 22358)\, f_{2n-1} - (45013\omega + 33683)\, f_n^2 + (14272\omega + 12928)\cdot(-1)^{n+1}] +$
$+ f_{n+3}\cdot[(1453\omega + 12163)\, f_{2n} + (4368\omega + 14128)\, f_{2n-1} - (22568\omega + 27523)\, f_n^2 +$
$(26448\omega + 24138)\cdot(-1)^{n+1}] = f_{n+2}\cdot[\omega\,(27923 f_{2n} + 30766 f_{2n-1}) + (20533 f_{2n} + 22358 f_{2n-1})$ -
$-\,(45013\omega + 33683)\, f_n^2 + (14272\omega + 12928)\cdot(-1)^{n+1}] + f_{n+3}\cdot[\omega\,(1453 f_{2n} + 4368 f_{2n-1}) +$
$+\,(12163 f_{2n} + 14128 f_{2n-1}) - (22568\omega + 27523)\, f_n^2 + (26448\omega + 24138)\cdot(-1)^{n+1}] =$
$f_{n+2}\cdot\left[\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533} - (45013\omega + 33683)\, f_n^2 + (14272\omega + 12928)\cdot(-1)^{n+1}\right] +$
$f_{n+3}\cdot\left[\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163} - (22568\omega + 27523)\, f_n^2 + (26448\omega + 24138)\cdot(-1)^{n+1}\right] =$
$f_{n+2}\cdot\left(\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533}\right) + f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163}\right) - f_n^2[\omega\,(45013 f_{n+2} + 22568 f_{n+3}) +$
$+\,(33683 f_{n+2} + 27523 f_{n+3})] + (-1)^{n+1}\cdot[\omega\,(1472 f_{n+2} + 26448 f_{n+3}) + (12982 f_{n+2} + 24138 f_{n+3})] =$
$f_{n+2}\cdot\left(\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533}\right) + f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163}\right) - f_n^2\cdot\left(\omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523}\right)$
$+ (-1)^{n+1}\cdot\left(\omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138}\right).$

Therefore, we obtained:

$$E\,(f_{n+1}, f_{n+4}, f_{n+7}) + E\,(f_n, f_{n+1}, f_{n+2}) + E\,(f_{n+3}, f_{n+4}, f_{n+5}) + E\,(f_{n+6}, f_{n+7}, f_{n+8}) +$$
$$E\,(\omega f_n, f_{n+5}, f_{n+7}) + E\,(f_{n+1}, f_{n+3}, \omega f_{n+8}) + E\,(f_{n+2}, f_{n+4}, \omega f_{n+6}) +$$
$$E\left(f_n, f_{n+4}, \omega^2 f_{n+8}\right) + E\left(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}\right) + E\left(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}\right) =$$
$$f_{n+2}\cdot\left(\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533}\right) + f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163}\right) - f_n^2\cdot\left(\omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523}\right)$$
$$+ (-1)^{n+1}\cdot\left(\omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138}\right). \qquad (5.25.)$$

Adding equalities (5.7), (5.15), (5.25), we have:

$$\eta(F_n) = 4a^2 h_{n+3}^{211,14}\cdot\left(h_{2n}^{84,135} - 2f_n^2\right) + 8h_{n+3}^{8,3}\cdot\left(h_{2n}^{12,20} - f_n^2\right) + a[f_{n+2}\left(\omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533}\right) +$$
$$f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163}\right) - f_n^2\left(\omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523}\right) - (-1)^n\left(\omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138}\right)].$$

□

Corollary 5.6. *Let F_n be the n th Fibonacci symbol element. Let ω be a primitive root of order 3 of unity and let $K = \mathbb{Q}(\omega)$ be the cyclotomic field . Then, the norm of F_n is*

$$\eta(F_n) = f_{n+2}\left(\omega h_{2n}^{30766,27923} + h_{2n}^{26822,27753}\right) + f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{19120,20203}\right) - f_n^2\cdot(\omega h_{n+3}^{45013,22563} +$$

$$+h_{n+3}^{33835,27659}\right)+(-1)^{n+1}\left(\omega h_{n+3}^{1472,26448}+h_{n+3}^{12982,24138}\right).$$

Proof. From the relation (5.7), we know that

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)=4\left(11f_{n+2}+14f_{n+3}\right)\cdot\left(h_{2n}^{84,135}-2f_n^2\right).$$

Therefore, we obtain

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)=f_{n+2}\left(h_{2n}^{3696,5940}-88f_n^2\right)+f_{n+3}\left(h_{2n}^{4704,7560}-112f_n^2\right). \tag{5.26.}$$

From the relation (5.15), we know that

$$E\left(f_n,f_{n+3},f_{n+6}\right)=8\left(8f_{n+2}+3f_{n+3}\right)\cdot\left(h_{2n}^{12,20}-f_n^2\right).$$

Then, we obtain

$$E\left(f_n,f_{n+3},f_{n+6}\right)=f_{n+2}\left(h_{2n}^{768,1280}-64f_n^2\right)+f_{n+3}\left(h_{2n}^{288,480}-24f_n^2\right). \tag{5.27.}$$

Adding equalities (5.26) and (5.27), we have:

$$E\left(f_{n+2},f_{n+5},f_{n+8}\right)+E\left(f_n,f_{n+3},f_{n+6}\right)=f_{n+2}\left(h_{2n}^{3696,5940}-88f_n^2\right)+$$

$$+f_{n+3}\left(h_{2n}^{4704,7560}-112f_n^2\right)+f_{n+2}\left(h_{2n}^{768,1280}-64f_n^2\right)+f_{n+3}\left(h_{2n}^{288,480}-24f_n^2\right)=$$

$$=f_{n+2}\left(h_{2n}^{3696,5940}+h_{2n}^{768,1280}\right)+f_{n+3}\left(h_{2n}^{4704,7560}+h_{2n}^{288,480}\right)-f_n^2\left(152f_{n+2}+136f_{n+3}\right)=$$

$$=f_{n+2}\left(h_{2n}^{3696,5940}+h_{2n}^{768,1280}\right)+f_{n+3}\left(h_{2n}^{4704,7560}+h_{2n}^{288,480}\right)-f_n^2h_{n+3}^{152,136}=$$

$$=f_{n+2}h_{2n}^{4464,7220}+f_{n+3}h_{2n}^{4992,8040}-f_n^2h_{n+3}^{152,136}.$$

It results

$$\eta(F_n)=f_{n+2}h_{2n}^{4464,7220}+f_{n+3}h^{4992,8040}-f_n^2h_{n+3}^{152,136}+f_{n+2}\left(\omega h_{2n}^{30766,27923}+h_{2n}^{22358,20533}\right)+f_{n+3}\cdot$$

$$\left(\omega h_{2n}^{4368,1453}+h_{2n}^{14128,12163}\right)-f_n^2\left(\omega h_{n+3}^{45013,22563}+h_{n+3}^{33683,27523}\right)-(-1)^n\left(\omega h_{n+3}^{1472,26448}+h_{n+3}^{12982,24138}\right).$$

Therefore

$$\eta(F_n)=f_{n+2}\left(\omega h_{2n}^{30766,27923}+h_{2n}^{26822,27753}\right)+f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453}+h_{2n}^{19120,20203}\right)-f_n^2\cdot(\omega h_{n+3}^{45013,22563}+$$

$$+h_{n+3}^{33835,27659})+(-1)^{n+1}\left(\omega h_{n+3}^{1472,26448}+h_{n+3}^{12982,24138}\right).$$

In conclusion, although indices of the top of generalized Fibonacci numbers, the expressions of the norm $\eta(F_n)$ from Proposition 5.5 and from Corollary 5.6 are very large. As we can see, these expressions are much shorter than the formula founded in [Pi; 82] and, in addition, the powers of Fibonacci numbers in these expressions are 1 or 2.$\square$

Let $A = \left(\frac{1,1}{\mathbb{Q},\omega}\right)$ be a symbol algebra of degree 3. Using the norm form given in the Corollary 5.6, we obtain:
$\eta\left(F_n\right) = f_{n+2}\left(\omega h_{2n}^{30766,27923} + h_{2n}^{26822,27753}\right) + f_{n+3}\cdot\left(\omega h_{2n}^{4368,1453} + h_{2n}^{19120,20203}\right) -$
-$f_n^2\cdot\left(\omega h_{n+3}^{45013,22563} + h_{n+3}^{33835,27659}\right) + (-1)^{n+1}\left(\omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138}\right) =$
$= \omega\left(f_{n+2}h_{2n}^{30766,27923} + f_{n+3}h_{2n}^{4368,1453} - f_n^2 h_{n+3}^{45013,22563} + (-1)^{n+1} h_{n+3}^{1472,26448}\right) +$
$+\left(f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2 h_{n+3}^{33835,27659} + (-1)^{n+1} h_{n+3}^{12982,24138}\right).$
If $\eta\left(F_n\right) \neq 0$, we know that the element F_n is invertible. From relation $\eta\left(F_n\right) \neq 0$, it results

$$f_{n+2}h_{2n}^{30766,27923} + f_{n+3}h_{2n}^{4368,1453} - f_n^2 h_{n+3}^{45013,22563} + (-1)^{n+1} h_{n+3}^{1472,26448} \neq 0 \tag{5.28.}$$

or

$$f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2 h_{n+3}^{33835,27659} + (-1)^{n+1} h_{n+3}^{12982,24138} \neq 0 \tag{5.29.}$$

Using relation (5.1) ,we obtain that $\lim\limits_{n\to\infty}\frac{f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203}(-1)^{n+1}h_{n+3}^{12982,24138}}{f_n^2 h_{n+3}^{33835,27659}} = \frac{\sqrt{5}\cdot 20203}{27659} = L \approx 1,62$. Denoting
$T_1 = f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} + (-1)^{n+1} h_{n+3}^{12982,24138}$ and
$T_2 = f_n^2 h_{n+3}^{33835,27659}$, it results that for all $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, we have $\left|\frac{T_1}{T_2} - L\right| < \varepsilon$. We obtain the inequalities:

$$-\varepsilon + L < \frac{T_1}{T_2} < \varepsilon + L.$$

For a good choise of ε, we have that $\frac{T_1}{T_2} > 1$, therefore

$$f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2 h_{n+3}^{33835,27659} + (-1)^{n+1} h_{n+3}^{12982,24138} > 0,$$

for all $n \geq n_\varepsilon$. We obtain that $\eta\left(F_n\right) \neq 0$, for all $n \geq n_\varepsilon$.

Then we prove the following result:

Theorem 5.7. *Let* $S = \left(\frac{1,1}{\mathbb{Q},\omega}\right)$ *be a symbol algebra of degree* 3. *Then there is the natural number* $n_\varepsilon \in \mathbb{N}$ *such that for all* $n \geq n_\varepsilon$ *the element* F_n *is an invertible element, where* F_n *is a Fibonacci symbol element.* $\Box$

Conclusions. In this paper, we studied some properties of the matrix representation of symbol algebras of degree 3. Using some properties of left matrix representations, we solved equations with coefficients in these algebras and we found an octonion non-division algebra in a symbol algebra of degree three. The study of symbol algebras of degree three involves very complicated calculus and, usually, can be hard to find examples for some notions. We introduced the Fibonacci symbol elements , we gave an easier expresion of reduced norm of Fibonacci symbol elements, and, from this formula, we found examples of many invertible elements. Using some ideas from this paper, we expect to obtain some interesting new results in a further research.

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